In this note, we present two methods for solving linear constant coefficient ordinary differential equations (ODEs). The two methods are the complex frequency domain method based on Laplace transforms and the time domain method, respectively.

**Formulation of Linear Constant Coefficient ODEs**

Let us first formulate the linear constant coefficient ODEs we will consider in the sequel. In control systems, we often encounter systems described by linear ODEs with constant coefficients of the following form

\[
a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u, \quad n \geq m, a_n \neq 0
\]  

(1)

where \( y \) is the output of the system and \( u \) is the input. For real systems, \( n \geq m \) is typically satisfied; we will adopt this assumption hereafter. A system described by (1) is called an \( n \)-th order system.

In this note, we are interested in determining the system’s response \( y(t) \) for \( t > 0 \) by solving (1). In order to do so, we must know \( n \) initial conditions \( y(0), \dot{y}(0), \ldots, y^{(n-1)}(0) \) and the input \( u(t), t \geq 0 \); most texts on signals and control adopt such initial conditions.

**Remark 1** We emphasize that in most texts, when initial conditions \( y(0), \dot{y}(0), \ldots, y^{(n-1)}(0) \) are given, what is actually meant is that \( y(0^-), \dot{y}(0^-), \ldots, y^{(n-1)}(0^-) \) are given (so do not be surprised to see such slight abuse of notation). In the study of control systems, when we adopt \( y(0^-), \dot{y}(0^-), \ldots, y^{(n-1)}(0^-) \) as initial conditions, the Laplace transform method can be directly applied to solve linear constant coefficient ODEs (note that the Lapalce transform is defined by integration over the interval between 0\(^-\) and \( \infty \)). However, when using the time domain method, as we will see in the sequel, in order to determine the coefficients for the terms in the complete response, we need to use the conditions \( y(0^+), \dot{y}(0^+), \ldots, y^{(n-1)}(0^+) \), which in general may not be the same as \( y(0^-), \dot{y}(0^-), \ldots, y^{(n-1)}(0^-) \). Possible discontinuities of \( y, \dot{y}, \ldots, y^{(n-1)} \) can take place at \( t = 0 \) due to the effects of impulses or jumps of the input at \( t = 0 \). In our subsequent study of the time domain method, we will present methods on how to compute \( y(0^+), \dot{y}(0^+), \ldots, y^{(n-1)}(0^+) \) from the given \( y(0^-), \dot{y}(0^-), \ldots, y^{(n-1)}(0^-) \).

\[ \square \]

**Solution Method by Using Laplace Transform**

One convenient solution method for linear constant coefficient ODEs is the complex frequency domain method by using Laplace transforms. Such a method yields an easy way to obtain the complete solution. By applying the Laplace transform on both sides of (1), we can convert the ODE in time domain into an algebraic equation in complex frequency domain involving the Laplace transforms of the output and the input. The general procedure is as follows.

**Procedure for the Complex Frequency Domain Method:**

1. Take the Laplace transform on both sides of (1) to convert the ODE into an algebraic equation in \( s \) and obtain the expression for the Laplace transform of \( Y(s) \) by rearranging the algebraic equation. When taking the Laplace transforms, be sure to take into consideration the initial conditions and apply the real differentiation theorem.

2. Take the inverse Laplace transform of \( Y(s) \) to find the time domain response \( y(t), t > 0 \).

Let us use the following examples to illustrate this method.
**Example 1** Consider the ODE
\[ \ddot{y} + 5\dot{y} + 6y = u \]  
where the initial conditions are \( y(0^-) = 2, \dot{y}(0^-) = -1 \). Applying the Laplace transform on both sides of (2), we obtain
\[ [s^2Y(s) - sy(0^-) - \dot{y}(0^-)] + 5[sY(s) - y(0^-)] + 6Y(s) = U(s). \]
Solving (3) for \( Y(s) \), we have
\[ Y(s) = \frac{(s+5)y(0^-)+\dot{y}(0^-)}{s^2+5s+6} + \frac{1}{s^2+5s+6}U(s). \]

It should be noted that on the righthand side of (4), the first term \( \frac{(s+5)y(0^-)+\dot{y}(0^-)}{s^2+5s+6} \) only depends on the initial conditions but not on the external input, and hence is called zero-input response\(^1\) and denoted as \( Y'_{zp}(s) \). The second term \( \frac{1}{s^2+5s+6}U(s) \) only depends on the external input but not on the initial conditions, and hence is called zero-state response\(^2\) and denoted as \( Y_{zs}(s) \). The complete response \( Y(s) \) is the sum of \( Y'_{zp}(s) \) and \( Y_{zs}(s) \).

To determine the complete response \( y(t) \), \( t > 0 \), we only need to take the inverse Laplace transform of \( Y(s) \). For example, if \( u(t) = t^2 \), \( t \geq 0 \), (i.e., \( U(s) = \frac{2}{s^3} \)), we have (the partial fraction expansion can be easily obtained by hand computation or by the Matlab command **residue**)
\[ y(t) = \mathcal{L}^{-1}\left\{ \frac{2s^4 + 9s^3 + 2}{(s^2 + 5s + 6)s^3} \right\} = \mathcal{L}^{-1}\left\{ \frac{4.7500}{s^2 + 5s + 6} + \frac{-2.9259}{s + 2} + \frac{0.1759}{s + 3} + \frac{-0.2778}{s^2} + \frac{0.3333}{s^3} \right\} = 4.7500e^{-2t} - 2.9259e^{-3t} + 0.1759 - 0.2778t + 0.1667t^2, \quad t > 0. \]

As another example, if \( u(t) = e^t \), \( t \geq 0 \), (i.e., \( U(s) = \frac{1}{s-1} \)), we have
\[ y(t) = \mathcal{L}^{-1}\left\{ \frac{2s^2 + 7s - 8}{(s^2 + 5s + 6)(s - 1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{2s^2 + 7s - 8}{(s^2 + 5s + 6)(s - 1)} \right\} = 4.6667e^{-2t} - 2.7500e^{-3t} + 0.0833e^t, \quad t > 0. \]

**Example 2** Consider the ODE
\[ 2\ddot{y} + 3\dot{y} + y = \dot{u} \]  
where the initial conditions are \( y(0^-) = 0, \dot{y}(0^-) = 0 \). Applying the Laplace transform on both sides of (7), we obtain (assume \( u(t) = 0 \) for \( t < 0 \) so \( u(0^-) = 0 \) and hence the righthand side is \( sU(s) \))
\[ 2[s^2Y(s) - sy(0^-) - \dot{y}(0^-)] + 3[sY(s) - y(0^-)] + Y(s) = sU(s). \]
Solving (8) for \( Y(s) \), we have the complete response
\[ Y(s) = \frac{(2s + 3)y(0^-) + 2\dot{y}(0^-)}{2s^2 + 3s + 1} + \frac{sU(s)}{2s^2 + 3s + 1} = \frac{sU(s)}{2s^2 + 3s + 1}. \]

\(^1\)In general, a zero-input response is defined to be the response when the input is 0 for \( t \geq 0 \). It is therefore only dependent on the initial conditions. The zero-input response of a linear constant coefficient ODE is linear with respect to the initial conditions at \( t = 0^- \).

\(^2\)In general, a zero-state response is defined to be the response when the initial conditions are all 0 at \( t = 0^- \). It is therefore only dependent on the external input. The zero-state response of a linear constant coefficient ODE is linear with respect to the input.
Note that due to the zero initial conditions, in this example the complete response is equal to the zero-state response.

If \( u(t) \) is the unit step input \( 1(t) \), then we have

\[
y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{2s^2 + 3s + 1} \right\} = \mathcal{L}^{-1}\left\{ \frac{-1}{s + 1} + \frac{1}{s + 0.5} \right\} = e^{-t} + e^{-0.5t}, \quad t > 0.
\] (10)

In view of (10), it is interesting to note that for this example \( y(0^+) = 0 \), but \( \dot{y}(0^+) = \frac{4}{\pi}(e^{-t} + e^{-0.5t}) \) \bigg|_{t=0^+} = e^{-t} - 0.5e^{-0.5t} \bigg|_{t=0^+} = 0.5 \neq \dot{y}(0^-) \), i.e., there is a discontinuity of \( \dot{y} \) at \( t = 0 \). □

**Time Domain Solution Method**

Another solution method for linear constant coefficient ODEs is the time domain method, which is also called the method of undetermined coefficients. It is also referred to as the classic method. The idea behind such a method is that the complete solution \( y(t) \) can be expressed as the sum of two solutions

\[
y(t) = y_h(t) + y_p(t)
\] (11)

where \( y_h(t) \) is called the homogeneous solution (or also called the complementary solution) and \( y_p(t) \) is called the particular solution. If we regard \( y(t) \) as the response of a system, \( y_h(t) \) is usually called the natural response and \( y_p(t) \) the forced response.

**Remark 2** To solve (1), we first determine the forms of \( y_h(t) \) and \( y_p(t) \). \( y_h(t) \) typically contains some undetermined coefficients. We can then use the initial conditions \( y(0^+), \dot{y}(0^+), \ldots, y^{(n-1)}(0^+) \) to determine the values of these coefficients. This explains why the method is also called the method of undetermined coefficient. However, it should be emphasized that the initial conditions \( y(0^+), \dot{y}(0^+), \ldots, y^{(n-1)}(0^+) \) (NOT \( y(0^-), \dot{y}(0^-), \ldots, y^{(n-1)}(0^-) \)) should be used to determine these coefficients, even though only the conditions \( y(0^-), \dot{y}(0^-), \ldots, y^{(n-1)}(0^-) \) are given in the original ODE formulation. Hence, we need to convert the conditions \( y(0^-), \dot{y}(0^-), \ldots, y^{(n-1)}(0^-) \) to the conditions \( y(0^+), \dot{y}(0^+), \ldots, y^{(n-1)}(0^+) \) before proceeding to determine the unknown coefficients. □

The general procedure of the time domain method is as follows.

**Procedure for the Time Domain Method:**

1. Convert the conditions \( y(0^-), \dot{y}(0^-), \ldots, y^{(n-1)}(0^-) \) to the conditions \( y(0^+), \dot{y}(0^+), \ldots, y^{(n-1)}(0^+) \).
2. Determine the general form of the homogeneous solution \( y_h(t) \) which may contain undetermined coefficients.
3. Determine the particular solution \( y_p(t) \).
4. Use the conditions \( y(0^+), \dot{y}(0^+), \ldots, y^{(n-1)}(0^+) \) to determine the coefficients in the complete solution \( y(t) = y_h(t) + y_p(t) \). □

In the following, let us first study the steps (2) and (3) in the above procedure. Then we will discuss more details on the step (1).

**Homogeneous Solution**

The homogeneous solution for (1) is the solution \( y_h(t) \) to the corresponding homogeneous differential equation (which is obtained by setting the righthand side of (1) to 0)

\[
a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = 0.
\] (12)
To determine the form of $y_h(t)$, we simply assume that it is a combination of functions of the form $Ce^{\lambda t}$. Let $y_h(t) = Ce^{\lambda t}$ and substitute it into (12), we have

$$a_n C \alpha^n e^{\lambda t} + a_{n-1} C \lambda^{n-1} e^{\lambda t} + \cdots + a_1 C \lambda e^{\lambda t} + a_0 C e^{\lambda t} = 0.$$  

This leads to

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0. \quad (13)$$

The lefthand side of (13) is called the characteristic polynomial $f(\lambda)$ of the ODE (1).

In general, if the characteristic polynomial of an ODE has $n$ distinct roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ (they may be complex$^3$), then

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \cdots + C_n e^{\lambda_n t} \quad (14)$$

where $C_1, C_2, \cdots, C_n$ are undetermined coefficients (they may be complex$^4$). We call each $e^{\lambda_k t}$ $(k = 1, 2, \cdots, n)$ a mode of the system (1).

**Example 3** Consider the ODE

$$\ddot{y} + 5 \dot{y} + 6y = u \quad (15)$$

as in Example 1. The characteristic polynomial is $f(\lambda) = \lambda^2 + 5\lambda + 6$ and it has two roots $\lambda_1 = -2$ and $\lambda_2 = -3$. Therefore the homogenous solution is of the form

$$y_h(t) = C_1 e^{-2t} + C_2 e^{-3t}. \quad (16)$$

**Example 4** Consider the ODE

$$2 \dddot{y} + 3 \dot{y} + y = \dot{u} \quad (17)$$

as in Example 2. The characteristic polynomial is $f(\lambda) = 2\lambda^2 + 3\lambda + 1$ and it has two roots $\lambda_1 = -1$ and $\lambda_2 = -0.5$. Therefore the homogenous solution is of the form

$$y_h(t) = C_1 e^{-t} + C_2 e^{-0.5t}. \quad (18)$$

In the case of repeated roots for the characteristic polynomial, the $y_h(t)$ in (14) should be modified. For example, if the root $\lambda_1$ is repeated $r$ times, then in (14) the part corresponding to $\lambda_1$ should be

$$C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t} + \cdots + C_r t^{r-2} e^{\lambda_1 t} + C_r t^{r-1} e^{\lambda_1 t}. \quad (19)$$

In (19), we also call each of the terms $e^{\lambda_1 t}, t e^{\lambda_1 t}, \cdots, t^{r-2} e^{\lambda_1 t}, t^{r-1} e^{\lambda_1 t}$ a mode of the system (1).

**Example 5** Consider the ODE

$$y^{(3)} + 7 \dddot{y} + 15 \dot{y} + 9y = 2 \dot{u} + u. \quad (20)$$

For this ODE, the characteristic polynomial is $f(\lambda) = \lambda^3 + 7\lambda^2 + 15\lambda + 9$ and it has roots $\lambda_1 = -1$ and $\lambda_{2,3} = -3$. Therefore the homogenous solution is of the form

$$y_h(t) = C_1 e^{-t} + C_2 e^{-3t} + C_3 t e^{-3t}. \quad (21)$$

---

$^3$For characteristic polynomials with real coefficient, complex roots always appear as complex conjugate pairs.

$^4$For a pair of complex conjugate roots, the corresponding coefficients are also complex conjugate numbers.
Particular Solution

The form of the particular solution $y_p(t)$ for (1) depends on the form of the input $u(t)$ for $t > 0$. Let us use some examples to illustrate how to obtain $y_p(t)$.

**Example 6** Consider the ODE

$$\ddot{y} + 5\dot{y} + 6y = u$$  \hspace{1cm} (22)

as in Example 1.

If we are given $u(t) = t^2$, $t \geq 0$, then we have $u(t) = t^2$ for $t > 0$ in this case. If we substitute $u(t) = t^2$ into the righthand side of (22) we obtain a polynomial expression in $t$ on the right. Hence, in order to balance the equation, we choose the form of $y_p(t)$ to be

$$y_p(t) = P_2t^2 + P_1t + P_0$$  \hspace{1cm} (23)

where $P_2$, $P_1$, and $P_0$ are constants to be determined. Substituting (23) into (22), we have

$$2P_2 + 5(2P_2t + P_1) + 6(P_2t^2 + P_1t + P_0) = t^2,$$

i.e.,

$$6P_2t^2 + (10P_2 + 6P_1)t + (2P_2 + 5P_1 + 6P_0) = t^2. \hspace{1cm} (24)$$

In (24) the coefficients of the polynomials in $t$ must be equal on both sides. This leads to

$$\begin{cases} 
6P_2 = 1 \\
10P_2 + 6P_1 = 0 \\
2P_2 + 5P_1 + 6P_0 = 0
\end{cases}$$

which consequently gives us $P_2 = 0.1667$, $P_1 = -0.2778$, and $P_0 = 0.1759$. Hence

$$y_p(t) = 0.1667t^2 - 0.2778t + 0.1759. \hspace{1cm} (25)$$

On the other hand, if we are given $u(t) = e^t$, $t \geq 0$, then we have $u(t) = e^t$ for $t > 0$. If we substitute $u(t) = e^t$ into the righthand side of (22) we obtain an expression containing $e^t$ on the right. Hence, in order to balance the equation, we choose the form of $y_p(t)$ to be

$$y_p(t) = Pe^t$$  \hspace{1cm} (26)

where $P$ is a constant to be determined. Substituting (26) into (22), we have

$$Pe^t + 5Pe^t + 6Pe^t = e^t,$$

which leads to $P = \frac{1}{12} = 0.0833$. Hence

$$y_p(t) = 0.0833e^t. \hspace{1cm} (27)$$

**Example 7** Consider the ODE

$$2\ddot{y} + 3\dot{y} + y = \dot{u}$$  \hspace{1cm} (28)

as in Example 2.

\footnote{This argument may appear redundant at the first glance. However, we point out that in the case of $u(t) = \delta(t)$, this argument reveals the important information that $u(t) = 0$ for $t > 0$ and hence leads to the particular solution $y_p(t) = 0$.}
If we are given \( u(t) = 1(t) \), then we have \( u(t) = 1 \) for \( t > 0 \). If we substitute \( u(t) = 1 \) into the righthand side of (28) we obtain a constant expression on the right. Hence, in order to balance the equation, we choose the form of \( y_p(t) \) to be

\[
y_p(t) = P
\]

where \( P \) is a constant to be determined. Substituting (29) into (28), we obtain \( P = 0 \). Hence

\[
y_p(t) = 0.
\]

The following table presents the forms of particular solutions for some typical inputs encountered in practice.

<table>
<thead>
<tr>
<th>( u(t), \ t &gt; 0 )</th>
<th>( y_p(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>constant ( C \neq 0 )</td>
<td>( P )</td>
</tr>
<tr>
<td>( t^r )</td>
<td>( P_r t^r + P_{r-1} t^{r-1} + \cdots + P_1 t + P_0 )</td>
</tr>
<tr>
<td>( e^{\alpha t} )</td>
<td>( Pe^{\alpha t} )</td>
</tr>
<tr>
<td>( \cos \omega t ) or ( \sin \omega t )</td>
<td>( P_1 \cos \omega t + P_2 \sin \omega t )</td>
</tr>
<tr>
<td>( t^r e^{\alpha t} \cos \omega t ) or ( t^r e^{\alpha t} \sin \omega t )</td>
<td>( (P_r t^r + \cdots + P_1 t + P_0) e^{\alpha t} \cos \omega t + (Q_r t^r + \cdots + Q_1 t + Q_0) e^{\alpha t} \sin \omega t )</td>
</tr>
</tbody>
</table>

**Initial Conditions at \( t = 0^+ \)**

Once the homogeneous and particular solution forms are obtained, we can use the initial conditions at \( t = 0^+ \) to determine the unknown coefficients (in the homogeneous solution). However, in the formulation of most ODEs, initial conditions are typically given at \( t = 0^- \). In the following, let us discuss two methods to convert the given initial conditions at \( t = 0^- \) to initial conditions at \( t = 0^+ \).

Before proceeding, let us first introduce some concepts for singularity functions. By *singularity functions*, we mean the unit ramp, the unit step, the unit impulse, the (generalized) derivative of unit impulse (called unit doublet), the (generalized) derivative of unit doublet (called unit triplet), etc. In other words, singularity functions are the family of functions obtained by integrals or (generalized) derivatives of the unit impulse functions. In the following, we say that one singularity function is of higher order of singularity than another singularity function if it can be obtained by taking (generalized) derivatives (maybe several times) of another function\(^6\).

Now let us present the two methods to convert the given initial conditions at \( t = 0^- \) to initial conditions at \( t = 0^+ \) through some examples.

**Method 1: The Method of Direct Inspection**

The method of direct inspection is based on the idea of balancing the highest order singularity on both sides of the ODE. Since the highest order singularity can only be generated by the highest order derivative terms on both sides of the ODE\(^7\), we only need to focus on these terms. Let us use an example to illustrate this method.

\(^{6}\)To simplify the matter, we may refer to the unit impulse as a singularity of order 0. Any singularity that can be obtained by taking (generalized) derivatives of the unit impulse is of positive order, e.g., a doublet is a singularity of order 1, a triplet is of order 2, etc. Any singularity that can be obtained by integrations of the unit impulse is of negative order, e.g., a unit step is a singularity of order \(-1\), a unit ramp is of order \(-2\), a unit parabolic \(t^2/2\), \( t \geq 0 \) is of order \(-3\), etc. It is worth noting that any singularity of order \(-2\) or less is continuous at \( t = 0 \).

\(^{7}\)This is intuitively true, since otherwise if the highest order singularity is present in derivative terms other than the highest order derivative term, then an even higher order singularity function would be generated by the highest order derivative term. This leads to a contradiction.
Example 8 Let us again consider the ODE studied in Example 1
\[ \ddot{y} + 5\dot{y} + 6y = u \]  

where the initial conditions are \( y(0^-) = 2, \dot{y}(0^-) = -1. \)

If \( u(t) = t^2, t \geq 0, \) then the highest order singularity present on the righthand side of (31) is of order −3. Hence the highest order singularity contained in \( \ddot{y} \) is of order −3 (of course, \( \ddot{y} \) may contain other lower order singularities). Consequently \( \dot{y} \) can only contain singularities up to order −4, and \( y \) can only contain singularities up to order −5. The implies that both \( y \) and \( \dot{y} \) should be continuous at 0, i.e.,

\[ y(0^+) = y(0^-) = 2, \quad \dot{y}(0^+) = \dot{y}(0^-) = -1. \]

Combining the homogeneous and particular solutions obtained in Examples 3 and 6, the complete response \( y(t) \) can be obtained as
\[ y(t) = y_h(t) + y_p(t) = C_1e^{-2t} + C_2e^{-3t} + 0.1667t^2 - 0.2778t + 0.1759, \quad t > 0. \]  

By substituting \( t = 0^+ \) into (33) and using the initial values in (32), we have

\[
\begin{cases}
    y(0^+) = C_1 + C_2 + 0.1759 = 2, \\
    \dot{y}(0^+) = -2C_1 - 3C_2 - 0.2778 = -1.
\end{cases}
\]

Solving the above equations, we obtain \( C_1 = 4.7501 \) and \( C_2 = -2.9260. \) Hence
\[ y(t) = 4.7501e^{-2t} - 2.9260e^{-3t} + 0.1667t^2 - 0.2778t + 0.1759, \quad t > 0, \]  

which is the same response as in (5) (except for slight float point calculation errors in the coefficients).

Next consider the case of \( u(t) = e^t, t \geq 0 \) (or equivalently \( u(t) = e^t1(t) \)), the highest order singularity present on the righthand side of (31) is of order −1. Hence the highest order singularity contained in \( \ddot{y} \) is of order −1. Consequently \( \dot{y} \) can only contain singularities up to order −2, and \( y \) can only contain singularities up to order −3. The implies that both \( y \) and \( \dot{y} \) should be continuous at 0, i.e.,

\[ y(0^+) = y(0^-) = 2, \quad \dot{y}(0^+) = \dot{y}(0^-) = -1. \]

Combining the homogeneous and particular solutions obtained in Examples 3 and 6, the complete response \( y(t) \) can be obtained as
\[ y(t) = y_h(t) + y_p(t) = C_1e^{-2t} + C_2e^{-3t} + 0.0833e^t, \quad t > 0. \]  

By substituting \( t = 0^+ \) into (36) and using the initial values in (35), we have

\[
\begin{cases}
    y(0^+) = C_1 + C_2 + 0.0833 = 2, \\
    \dot{y}(0^+) = -2C_1 - 3C_2 + 0.0833 = -1.
\end{cases}
\]

Solving the above equations, we obtain \( C_1 = 4.6668 \) and \( C_2 = -2.7501. \) Hence
\[ y(t) = 4.6668e^{-2t} - 2.7501e^{-3t} + 0.0833e^t, \quad t > 0, \]  

which is the same response as in (6) (except for slight float point calculation errors in the coefficients). □

Example 9 Let us again consider the ODE studied in Example 2
\[ 2\ddot{y} + 3\dot{y} + y = \dot{u} \]  

where the initial conditions are \( y(0^-) = 0, \dot{y}(0^-) = 0. \)
In deriving (44) and (45), we use the property of the impulse function that $f \cdot \delta(t) = f(0)\delta(t)$ for any function $f(t)$ which is continuous at $t = 0$.

If $u(t) = 1(t)$, then the highest order singularity present on the righthand side of (38) is $\delta(t)$, which is of order 0. Hence the highest order singularity contained in $\dot{y}$ is $0.5\delta(t)$ (here the coefficient of $\delta(t)$ is 0.5 due to the coefficient 2 for $\dot{y}$ in (38)). Consequently the highest order singularity contained in $\ddot{y}$ is $0.5 \cdot 1(t)$ (which means that $\dot{y}$ has a jump of 0.5 at $t = 0$), which is of order 1; and $y$ contains singularities up to order $-2$. The implies that

$$y(0^+) = y(0^-) = 0,\quad \dot{y}(0^+) = \dot{y}(0^-) + 0.5 = 0.5.$$  \hspace{1cm} (39)

Combining the homogeneous and particular solutions obtained in Examples 4 and 7, the complete response $y(t)$ can be obtained as

$$y(t) = y_h(t) + y_p(t) = C_1 e^{-t} + C_2 e^{-0.5t},\quad t > 0.$$  \hspace{1cm} (40)

By substituting $t = 0^+$ into (40) and using the initial values in (39), we have

$$\begin{cases}
y(0^+) = C_1 + C_2 = 0,
\dot{y}(0^+) = -C_1 - 0.5C_2 = 0.
\end{cases}$$

Solving the above equations, we obtain $C_1 = -1$ and $C_2 = 1$. Hence

$$y(t) = -e^{-t} + e^{-0.5t},\quad t > 0,$$  \hspace{1cm} (41)

which is the same response as (10).

**Method 2: The Method of Balancing Singularity Functions (for Zero-State Response)**

The method of direct inspection is easy to apply in most cases but requires some intuition to determine the initial conditions at $t = 0^+$. A more systematic method for determining the response without first computing the initial conditions at $t = 0^+$ (here esp., zero-state response) is based on the balancing of singularity functions. We emphasize that this method can only be applied to cases with zero initial conditions at $t = 0^-$, i.e., to determine the zero-state response. However, in our subsequent discussion, we will extend this method to a general method for determining the complete response based on the separation of zero-input and zero-state responses. Here let us first illustrate this method for the ODE in Example 2.

**Example 10** Let us again consider the ODE studied in Example 2

$$2\ddot{y} + 3\dot{y} + y = \dot{u}$$  \hspace{1cm} (42)

where the initial conditions are $y(0^-) = 0$, $\dot{y}(0^-) = 0$, and $u(t) = 1(t)$.

In view of the homogeneous and particular solutions obtained in Examples 4 and 7, let us assume that the complete solution is

$$y(t) = (y_h(t) + y_p(t))1(t) = (C_1 e^{-t} + C_2 e^{-0.5t} + 0) \cdot 1(t),\quad t > 0.$$  \hspace{1cm} (43)

Here we multiply $(y_h(t) + y_p(t))$ by $1(t)$ to emphasize that the initial conditions at $t = 0^-$ are 0. This is a key step of the method (since we need the derivative of $1(t)$ when balancing the singularities on both sides of (42)). From (43), we have

$$\dot{y}(t) = (C_1 e^{-t} + C_2 e^{-0.5t})\delta(t) + (-C_1 e^{-t} - 0.5C_2 e^{-0.5t}) \cdot 1(t)$$

$$= (C_1 + C_2)\delta(t) + (-C_1 e^{-t} - 0.5C_2 e^{-0.5t}) \cdot 1(t),$$  \hspace{1cm} (44)

$$\ddot{y}(t) = (C_1 + C_2)\delta'(t) + (-C_1 e^{-t} - 0.5C_2 e^{-0.5t})\delta(t) + (C_1 e^{-t} + 0.25C_2 e^{-0.5t}) \cdot 1(t)$$

$$= (C_1 + C_2)\delta'(t) + (-C_1 - 0.5C_2)\delta(t) + (C_1 e^{-t} + 0.25C_2 e^{-0.5t}) \cdot 1(t).$$  \hspace{1cm} (45)

In deriving (44) and (45), we use the property of the impulse function that $f(t)\delta(t) = f(0)\delta(t)$ for any function $f(t)$ which is continuous at $t = 0$. 
Substituting \( u(t) = 1(t) \) and (43)-(45) into (42) and simplifying, we have

\[
(2C_1 + 2C_2)\delta'(t) + (C_1 + 2C_2)\delta(t) = \delta(t).
\]  

Balancing the coefficients for the singularities in the above equation, we have

\[
\begin{cases}
2C_1 + 2C_2 = 0, \\
C_1 + 2C_2 = 1,
\end{cases}
\]

which leads to \( C_1 = -1 \) and \( C_2 = 1 \). Hence

\[
y(t) = -e^{-t} + e^{-0.5t}, \quad t > 0,
\]

which is the same response as in (10). From the complete response in (47), we can verify that \( y(0^+) = 0 \) and \( \dot{y}(0^+) = 0.5 \). But Method 2 does not require that we first compute these initial conditions at \( t = 0^+ \), instead it computes the complete solution directly.

Comparing the zero-state response (47) and the forced response (30), we should note that the zero-state response of an ODE is in general not the same as its forced response (i.e., particular solution). The zero-state response is obtained by adding a weighted sum of the modes of the system and the forced response. \( \square \)

**Remark 3** Method 2 is especially useful in computing the unit impulse or unit step responses of a linear time invariant (LTI) system (an LTI system is typically described by a linear constant coefficient ODE). Since in deriving such responses, an implicit assumption is that all the initial conditions are 0 at \( t = 0^- \). \( \square \)

**A General Approach for Time Domain Solution Method — Separation of Zero-Input and Zero-State Responses**

Although the Method 2 above is more systematic than Method 1, yet it can only be applied to determining the zero-state response of an ODE. In the sequel, we will extend it to ODEs with nonzero initial conditions. We propose a general approach for solving ODEs in the time domain. The approach is based on the separation of a system’s complete response into zero-input and zero-state responses. First we point out that the complete response of a linear constant coefficient ODE can be decomposed as follows (for illustration purpose and simplicity of notations, we only present the response an ODE with distinct characteristic roots in (48) and (49); but it can be readily extended to the case of repeated characteristic roots).

\[
y(t) = \sum_{k=1}^{n} C_k e^{\lambda_k t} + y_p(t), \quad t > 0 \tag{48}
\]

natural response \( y_n(t) \)

\[
= \sum_{k=1}^{n} C_{zp_k} e^{\lambda_k t} + \sum_{k=1}^{n} C_{zs_k} e^{\lambda_k t} + y_p(t), \quad t > 0 \tag{49}
\]

zero-input response \( y_{zp}(t) \)

zero-state response \( y_{zs}(t) \)

**Remark 4** From (49), we note that the zero-input response \( y_{zp}(t) \) can be obtained as a weighted sum of the modes of the system. This is due to the following reasons: (i). In finding the zero-input response, since there is no external input, we do not have to consider the forced response; (ii). Since there is no external input, we can simply assume that the initial condition is continuous at 0 for the purpose of finding \( y_{zp}(t) \) (in other words, the coefficients \( C_{zp_k} \)'s can be determined by using the given initial conditions). \( \square \)

**Example 11** Consider the following ODE

\[
\ddot{y} + 4\dot{y} + 3y = \dot{u} + 2u \tag{50}
\]
where the initial conditions are \( y(0^-) = 1, \dot{y}(0^-) = 0 \), and \( u(t) = 1(t) \).

For this ODE, the characteristic polynomial is \( f(\lambda) = \lambda^2 + 4\lambda + 3 \) and it has two roots \( \lambda_1 = -1, \lambda_2 = -3 \). Therefore the homogeneous solution (i.e., the natural response) is of the form

\[
y_h(t) = C_1e^{-t} + C_2e^{-3t}.
\]

**Zero-Input Response:** To determine the zero-input response, we may simply assume

\[
y_{zp}(t) = C_{zp1}e^{-t} + C_{zp2}e^{-3t}, \quad t > 0.
\]

Since there is no external input, the initial conditions will be continuous at \( t = 0 \), i.e., \( y(0^+) = y(0^-) = 1 \), \( \dot{y}(0^+) = \dot{y}(0^-) = 0 \). By substituting \( t = 0^+ \) into (52) and using the initial conditions at \( t = 0^+ \), we have

\[
\begin{cases}
  y(0^+) = C_{zp1} + C_{zp2} = 1, \\
  \dot{y}(0^+) = -C_{zp1} - 3C_{zp2} = 0,
\end{cases}
\]

which leads to \( C_{zp1} = 1.5, C_{zp2} = -0.5 \). Hence

\[
y_{zp}(t) = 1.5e^{-t} - 0.5e^{-3t}, \quad t > 0.
\]

**Zero-State Response:** When the input is \( 1(t) \), we have \( u(t) = 1 \) for \( t > 0 \). We can then choose the form of the forced response (i.e., the particular solution) to be

\[
y_p(t) = P
\]

where \( P \) is a constant to be determined. Substituting (54) into (50), we have \( 3P = 2 \) and hence \( P = \frac{2}{3} = 0.6667 \). To apply the Method 2 mentioned above to determine the zero-state response, let us assume that

\[
y_{zs}(t) = (C_{zs1}e^{-t} + C_{zs2}e^{-3t} + 0.6667) \cdot 1(t), \quad t > 0.
\]

From (55), we have

\[
\begin{align*}
\dot{y}_{zs}(t) &= (C_{zs1}e^{-t} + C_{zs2}e^{-3t} + 0.6667)\delta(t) + (-C_{zs1}e^{-t} - 3C_{zs2}e^{-3t}) \cdot 1(t) \\
&= (C_{zs1} + C_{zs2} + 0.6667)\delta(t) + (-C_{zs1}e^{-t} - 3C_{zs2}e^{-3t}) \cdot 1(t),
\end{align*}
\]

\[
\begin{align*}
\ddot{y}_{zs}(t) &= (C_{zs1} + C_{zs2} + 0.6667)\delta'(t) + (-C_{zs1}e^{-t} - 3C_{zs2}e^{-3t})\delta(t) + (C_{zs1}e^{-t} + 9C_{zs2}e^{-3t}) \cdot 1(t) \\
&= (C_{zs1} + C_{zs2} + 0.6667)\delta'(t) + (-C_{zs1} - 3C_{zs2})\delta(t) + (C_{zs1}e^{-t} + 9C_{zs2}e^{-3t}) \cdot 1(t).
\end{align*}
\]

Substituting \( u(t) = 1(t) \) and (55)-(57) into (50) and simplifying, we have

\[
(C_{zs1} + C_{zs2} + 0.6667)\delta'(t) + (3C_{zs1} + C_{zs2} + 2.6667)\delta(t) = \delta(t).
\]

Balancing the coefficients for the singularities in the above equation, we have

\[
\begin{cases}
  C_{zs1} + C_{zs2} + 0.6667 = 0, \\
  3C_{zs1} + C_{zs2} + 2.6667 = 1,
\end{cases}
\]

which leads to \( C_{zs1} = -0.5 \) and \( C_{zs2} = -0.1667 \). Hence

\[
y_{zs}(t) = -0.5e^{-t} - 0.1667e^{-3t} + 0.6667, \quad t > 0.
\]

In view of the above results, the complete response of the ODE is

\[
y(t) = y_{zp}(t) + y_{zs}(t) = e^{-t} - 0.6667e^{-3t} + 0.6667, \quad t > 0.
\]
Verification of the Responses Using the Complex Frequency Domain Method: Let us verify the above responses by using the Laplace transform method. Applying the Laplace transform on both sides of (50), we obtain (assume $u(t) = 0$ for $t < 0$ so $u(0^-) = 0$ and hence the righthand side is $sU(s) + 2U(s)$)

$$[s^2Y(s) - sy(0^-) - \dot{y}(0^-)] + 4[sY(s) - y(0^-)] + 3Y(s) = sU(s) + 2U(s).$$

Solving (61) for $Y(s)$, we obtain the complete response

$$Y(s) = \frac{(s + 4)y(0^-) + \dot{y}(0^-)}{s^2 + 4s + 3} + \frac{s + 2}{s^2 + 4s + 3}U(s).$$

(62)

From (62), we have

$$y_{zp}(t) = \mathcal{L}^{-1}\left\{\frac{s + 4}{s^2 + 4s + 3}\right\} = \mathcal{L}^{-1}\left\{\frac{1.5}{s + 1} - \frac{0.5}{s + 3}\right\} = 1.5e^{-t} - 0.5e^{-3t}, \ t > 0,$$

(63)

$$y_{zs}(t) = \mathcal{L}^{-1}\left\{\frac{s + 2}{s(s^2 + 4s + 3)}\right\} = \mathcal{L}^{-1}\left\{\frac{0.6667}{s} + \frac{-0.5}{s + 1} + \frac{-0.1667}{s + 3}\right\} = 0.6667 - 0.5e^{-t} - 0.1667e^{-3t}, \ t > 0,$$

(64)

which are the same as those in (53) and (59).

Remark 5 Finally, we point out that in Example 11, we cannot directly assume that the zero-input response is $y_{zp}(t) = (C_{zp1}e^{-t} + C_{zp2}e^{-3t}) \cdot 1(t)$ since such an assumption will lead to $y_{zp}(0^-) = 0$ which is not correct according to the given condition $y(0^-) = 1$. Hence, in this case, we cannot directly add $y_{zp}(t)$ and $y_{zs}(t)$ to assume $y(t) = (C_{1}e^{-t} + C_{2}e^{-3t} + 0.6667) \cdot 1(t)$. Therefore we should be cautious that we cannot simply substitute $y(t) = (C_{1}e^{-t} + C_{2}e^{-3t} + 0.6667) \cdot 1(t)$ into (50) so as to determine the correct $C_1$ and $C_2$ values. Once again, we emphasize that only in the case of zero-state response can we substitute (55) into (50) and apply the Method 2 mentioned above to determine the zero-state response.

\[\text{Footnote: Note the difference between } y_{zp}(t) = C_{zp1}e^{-t} + C_{zp2}e^{-3t}, \ t > 0 \text{ and } y_{zp}(t) = (C_{zp1}e^{-t} + C_{zp2}e^{-3t}) \cdot 1(t). \text{ In the former case, we only explicitly specify the expression of } y_{zp}(t) \text{ for } t > 0 \text{ but does not alter the conditions at } t = 0^- \text{. However, in the latter case, we specify the expression of } y_{zp}(t) \text{ completely over } -\infty < t < \infty \text{ so that the initial conditions at } t = 0^- \text{ may be altered.}\]